Theorem. *Suppose that for $x \to a+$ we have either*

$$f(x) \to 0 \quad \textit{and} \quad g(x) \to 0 \tag{1}$$

or

$$g(x) \to +\infty \quad \textit{or} \quad g(x) \to -\infty. \tag{2}$$

Bounds for $\frac{f'}{g'}$ in an open interval $I = (a, \dots)$ are preserved unchanged for $\frac{f}{g}$ in the first case, and up to an arbitrary $\varepsilon > 0$ in a subinterval $I' = (a, \dots)$ in the second case.

Proof. Assume that

$$\frac{f'(x)}{g'(x)} \leqslant A \in \mathbb{R} \qquad (x \in I). \tag{3}$$

(For $\frac{f'(x)}{g'(x)} \geqslant B \in \mathbb{R}$ all inequalities are reversed.) In particular, $g'(x) \neq 0$ in I. Due to the intermediate value property of derivatives, g cannot change signs, say $g'(x) > 0$ in I. (In the opposite case, consider $-g$. Note that g is strictly increasing in I, and $g(x) \to +\infty$ cannot occur for $x \to a+$.) It follows that (3) is equivalent to $f' \leqslant Ag'$, implying that $Ag - f$ is increasing in I. Hence,

$$Ag(x_1) - f(x_1) \leqslant Ag(x_2) - f(x_2) \tag{4}$$

for all $x_1 < x_2$ in I. In the first case, $x_1 \to a+$ yields $0 \leqslant Ag(x_2) - f(x_2)$. Also, $g(x) > 0$ in I because g is strictly increasing with infimum 0. Hence $\frac{f(x_2)}{g(x_2)} \leqslant A$ for every $x_2 \in I$. In the second case, $g(x) < 0$ close to a because $g(x) \to -\infty$. Given $\varepsilon > 0$, (4) implies

$$A - \frac{f(x_1)}{g(x_1)} \geqslant \frac{Ag(x_2) - f(x_2)}{g(x_1)} > -\varepsilon$$

for all $x_1 \in I$ close enough to a. Therefore, $\frac{f(x_1)}{g(x_1)} < A + \varepsilon$ for all $x_1 \in I$ close enough to a.

Corollary. *If, for any kind of limit, we have* (1) *or* (2)*, and if* $\frac{f'}{g'} \to L \in \mathbb{R} \cup \{\pm\infty\}$*, then also* $\frac{f}{g} \to L$*.*

Proof for $x \to a+$. The property $\frac{f'}{g'} \to L$ is equivalent to one or two quantified inequalities for f'/g' (like: for every $\varepsilon > 0$ we have $L - \varepsilon \leqslant \frac{f'(x)}{g'(x)} \leqslant L + \varepsilon$ for all $x \in I$ close enough to a). These statements carry over to f/g either unchanged or with an extra $\varepsilon > 0$. Either way, they express $\frac{f}{g} \to L$.

For $x \to a-$: apply the previous to $f(-x)$ and $g(-x)$ (in the quotients, minus signs cancel); for $x \to a$: combine $x \to a-$ and $x \to a+$; for $x \to +\infty$: apply $x \to 0+$ to $f(1/x)$ and $g(1/x)$ (in the quotients, $-1/x^2$ cancels out); for $x \to -\infty$: apply $x \to 0+$ to $f(-1/x)$ and $g(-1/x)$.

Note. There are five types of limits (right-handed, left-handed, twosided, to $+\infty$, to $-\infty$), five indeterminate quotients (0/0 and, counting the signs, four types of ∞/∞), and three solutions (a real number, $+\infty$, $-\infty$). All in all, 75 cases for de l'Hospital.